Let $S \subseteq \mathbb{R}$. We call a real number x an *upper bound* for S if

$$(\forall s \in S)[x \geq s].$$

The *completeness axiom* states that if $S \subseteq \mathbb{R}$ is nonempty and has an upper bound, then it has a least upper bound, called the *supremum* of S. That is, there is a real number M such that:

1. $M \geq s$ for all $s \in S$;
2. For any upper bound x of S, we have $x \geq M$.

In fact, M is unique (see Problem 2). We use $\sup S$ to denote the[1] supremum of S.

Similarly, any $S \subseteq \mathbb{R}$ that is nonempty and bounded from below has a unique greatest lower bound called the *infimum*.

Problem 0
Briefly discuss the difference between *supremum* and *maximum*.

Problem 1
For each of the following sets, find the supremum/infimum or show that it doesn't exist. Which sets have maxima/minima?

1. $(-\infty, 0) \cup \{1\}$.
2. $\mathbb{Q} \cap (-\infty, \sqrt{2})$.
3. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Problem 2
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Show that the supremum of S is unique: if both M and N are least upper bounds for S, then $M = N$.

Problem 3
Let $S \subseteq \mathbb{R}$ have a maximum M. Show that $M = \sup S$.

Problem 4
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Show that $M = \sup S$ if and only if M is an upper bound for S and $(\forall \epsilon > 0)(\exists s \in S)[M < s + \epsilon]$. *Note: This is hard. Try taking the contrapositive of both implications.*

An analogous statement holds for infima: for any $S \subseteq \mathbb{R}$ nonempty and bounded from below, $m = \inf S$ if and only if $(\forall \epsilon > 0)(\exists s \in S)[s - \epsilon < m]$.

[1] Notice we can use "the" here as opposed to 'a': there is only one supremum due to uniqueness.

Here are some nice properties of sup (analogous properties hold for inf):

Problem 5
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above, and $c > 0$. Define

$$cS = \{cs : s \in S\}.$$

Show that $\sup cS = c \sup S$.

Problem 6
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Define

$$-S = \{-s : s \in S\}.$$

Show that $\inf -S = -\sup S$.

Problem 7
Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. We can define a new function $f + g : \mathbb{R} \to \mathbb{R}$ given by $(f + g)(x) = f(x) + g(x)$. We can also define $\sup h$, for any function $h : \mathbb{R} \to \mathbb{R}$, as:

$$\sup h = \sup\{h(x) : x \in \mathbb{R}\}$$

Prove that if $\sup f$ and $\sup g$ both exist, then

$$\sup(f + g) \leq \sup f + \sup g.$$

Give examples of f and g for which the above inequality is strict.